Constructing Linear Families from Parameter-Dependent Nonlinear Dynamics

Harry G. Kwatny and B.-C. Chang

***Abstract*— Generating families of linear models from nonlinear parameter-dependent equations requires explicit analytical characterization of the equilibrium surface. Doing so in terms of the original system parameters is generally not possible. Introducing an alternative parameterization, we propose an efficient method for computing local linear parameter-dependent families. Although local, these families can be constructed anywhere, specifically around bifurcation points where other methods fail.**

***Index Terms*— Linearization, nonlinear dynamics, parameter-dependent families, symbolic computing.**

I. Introduction

Control system designers are typically confronted with the requirement that a controller should provide satisfactory performance at several operating conditions. Accordingly, the plant to be controlled is often represented by a parameterized family of linear models where different parameter values correspond to different equilibrium points. These parameter-dependent linear models are frequently the basis for gain-scheduled, adaptive, and robust control system design [1]. In practice, however, the formulation of such models presents fundamental obstacles. In this paper we provide one approach to model construction when a nonlinear parameter dependent model is available.

Consider a parameterized family of nonlinear control systems in the form of

$$\begin{aligned} \dot{x} &= f(x, u, \mu) \\ y &= h(x, u, \mu) \end{aligned} \tag{1}$$

with $x \in R^n, u \in R^m, y \in R^m, \mu \in R^k$. If x_0, u_0, μ_0 corresponds to an equilibrium point, that is, $f(x_0, u_0, \mu_0) = 0$ and $y_0 = h(x_0, u_0, \mu_0) = 0$, the linear perturbation equations are

$$\begin{aligned} \delta\dot{x} &= \left[\frac{\partial f(x_0, u_0, \mu_0)}{\partial x}\right]\delta x + \left[\frac{\partial f(x_0, u_0, \mu_0)}{\partial u}\right]\delta u \\ \delta y &= \left[\frac{\partial h(x_0, u_0, \mu_0)}{\partial x}\right]\delta x + \left[\frac{\partial h(x_0, u_0, \mu_0)}{\partial u}\right]\delta u. \end{aligned} \tag{2}$$

Suppose, however, that we wish to construct a family of linear models in which the parameter μ is considered to be an independent variable. Then, in principle, we need to solve the algebraic equilibrium equations for $x_0(\mu), u_0(\mu)$ in order to obtain

$$\begin{aligned} \delta\dot{x} &= \left[\frac{\partial f(x_0(\mu), u_0(\mu), \mu)}{\partial x}\right]\delta x + \left[\frac{\partial f(x_0(\mu), u_0(\mu), \mu)}{\partial u}\right]\delta u \\ &= A(\mu)\delta x + B(\mu)\delta u \\ \delta y &= \left[\frac{\partial h(x_0(\mu), u_0(\mu), \mu)}{\partial x}\right]\delta x + \left[\frac{\partial h(x_0(\mu), u_0(\mu), \mu)}{\partial u}\right]\delta u \\ &= C(\mu)\delta x + D(\mu)\delta u. \end{aligned} \tag{3}$$

Manuscript received January 27, 1997. This work was supported by the NASA Langley Research Center under Contract NCC1–224.

The authors are with the Department of Mechanical Engineering and Mechanics, Drexel University, Philadelphia, PA 19104 USA (e-mail: hkwatny@coe.drexel.edu).

Publisher Item Identifier S 0018-9286(98)06120-0.

The need to characterize the dependence of equilibria on the parameters is the essential and difficult aspect of the linearization problem. The view that this can be accomplished by functions $x_0(\mu), u_0(\mu)$ is unsatisfactory. Assume that f and h are smooth functions and that equilibrium points are defined by the requirement

$$F(x, u, \mu) := \begin{bmatrix} f(x, u, \mu) \\ h(x, u, \mu) \end{bmatrix} = 0. \tag{4}$$

This relation generically defines a smooth k-dimensional manifold in R^{n+m+k}, called the *equilibrium set*, designated

$$ES = \left\{(x, u, \mu) \in R^{n+m+k} | F(x, u, \mu) = 0\right\}. \tag{5}$$

An example of such a manifold is shown in Fig. 1, adapted from [2]. Notice that the surface, although smooth, may have folds, and consequently we cannot expect globally valid functions $x(\mu), u(\mu)$ to define equilibria. Even local functions do not exist on neighborhoods of points along the "folds" of the surface. Yet these are probably the most interesting regions because these points correspond to static bifurcations such as stall in aircraft [2], [3].

In Section II we introduce the main idea—the identification of a new set of parameters that admit, in principle, a global definition of the equilibrium surface. This leads to a computational procedure that is described and illustrated with a simple example in Section III. In Section IV we apply the method to the longitudinal dynamics of an aircraft. The required symbolic calculations are performed using *Mathematica*.

II. A New Approach

We propose an alternative to the approach of computing $x(\mu), u(\mu)$ for the characterization of equilibria. Consider any point $(x_0, u_0, \mu_0) \in ES$. Our goal is to define a coordinate system on ES around (x_0, u_0, μ_0) and with the origin located at (x_0, u_0, μ_0). The k new coordinates s will replace the k parameters μ to provide a new parametric representation of equilibrium points and the associated linear dynamics in terms of s instead of μ (see Fig. 1). We will find a mapping $(x(s), u(s), \mu(s))\colon R^k \to R^{n+m+k}$ that defines the equilibrium manifold.

Consider (1) and concatenate all of the $n+m+k$ variables to define a single dependent variable $\bar{x} = (x, u, \mu)$. Let $\overline{F}(\overline{x}) := F(x, u, \mu)$ and $D_{\overline{x}}\overline{F}$ its Jacobian. Suppose we have that $\text{rank}[D_{\overline{x}}\overline{F}(\overline{x})] = n+m$ on the equilibrium set $ES = \{\overline{x} \in R^{n+m+k} | \overline{F}(\overline{x}) = 0\}$. This ensures that ES is a regular imbedded manifold of dimension k in R^{n+m+k} [4]. If $\overline{x}(s)$ satisfies $\overline{F}(\overline{x}(s)) = 0$, it must be true that

$$D_{\overline{x}}\overline{F}(\overline{x})\frac{\partial \overline{x}}{\partial s}ds = 0$$

for arbitrary ds at each point $\overline{x} \in ES$. Now, we can always find a basis $\{\gamma_1(\overline{x}) \cdots \gamma_k(\overline{x})\}$ for ker $D_{\overline{x}}\overline{F}$. Consequently, we must have

$$\frac{\partial \overline{x}(s)}{\partial s_i} \in \text{span}\{\gamma_1(\overline{x}) \cdots \gamma_k(\overline{x})\}, \qquad i = 1, \cdots, k. \tag{6}$$

The necessary conditions of (6) provide a means for computing the transformation.

A. Example 1

Consider the one-parameter dynamical system $\dot{x} = -x^2 - \mu$. Equilibria are defined by the equation $F(x, \mu) = x^2 + \mu = 0$. Thus, we compute

$$D_{\overline{x}}F = [2x \quad 1] \Rightarrow \gamma(x, \mu) = \begin{bmatrix} 1 \\ -2x \end{bmatrix} \Rightarrow \frac{d}{ds}\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} 1 \\ -2x \end{bmatrix}.$$

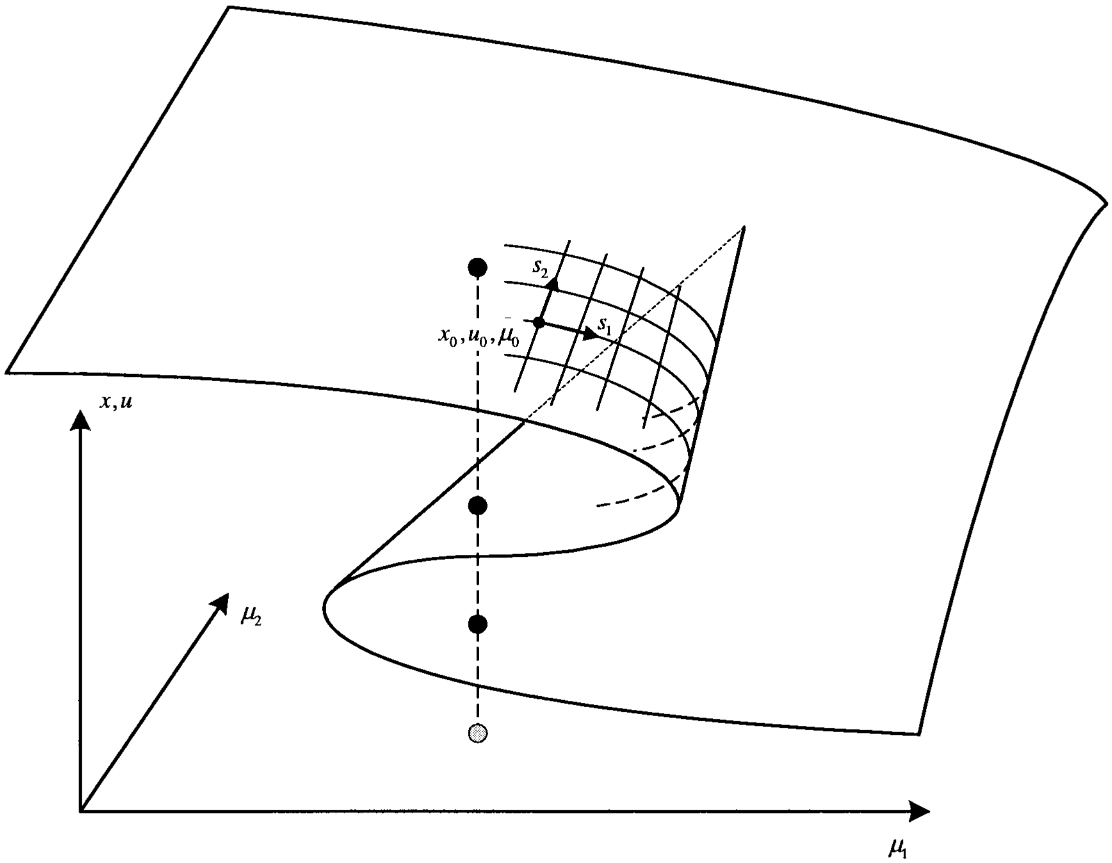

Fig. 1. A typical equilibrium manifold cannot be characterized by a function of the parameters μ_1, μ_2. In this illustration there are as many as three equilibrium points corresponding to a particular parameter value. A new set of coordinates (s_1, s_2) will replace the original parameters (μ_1, μ_2).

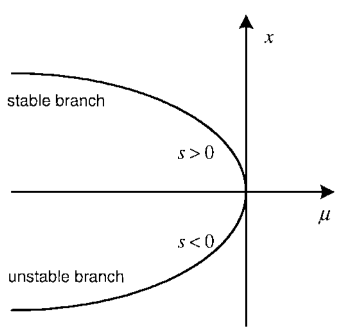

Fig. 2. The equilibrium manifold is the parabola illustrated in the x–μ space. The equilibria on the upper branch are stable. Those on the lower branch are unstable.

We can solve this last equation to obtain

$$x = s, \qquad \mu = -s^2.$$

These equations parametrically define the equilibrium surface. The perturbation dynamics are $\delta\dot{x} = [-2x_0]\delta x = [-2s]\delta x$. Fig. 2 summarizes these results.

III. Constructing Solutions

In the following paragraphs we describe and illustrate a general approach to the construction of the transformation $\overline{x}(s)$.

A. Composition of Flows

We can construct a mapping $\overline{x}(s)$ that satisfies (6) by solving the set of partial differential equations

$$\frac{\partial \overline{x}}{\partial s_i} = \gamma_i(\overline{x}), \qquad i = 1, \cdots, k. \tag{7}$$

Recall that the flow defined by the ith differential equation of (7) is the function $\phi_i^{s_i}(\overline{x})$ that satisfies

$$\frac{\partial \phi_i^{s_i}}{\partial s_i} = \gamma_i(\phi_i^{s_i}), \qquad \text{with } \phi_i^0(\overline{x}) = \overline{x}. \tag{8}$$

Now, we can state the following proposition.

Proposition: Suppose that F: $R^{N+k} \rightarrow R^N$ is a smooth (C^∞) mapping with rank $D_{\overline{x}}F = N$ on the set $ES = \{\overline{x} \in R^{N+k} | F(\overline{x}) = 0\}$. Then ES is a smooth, k-dimensional, regular manifold that is parametrically characterized by the mapping $\overline{x}$: $R^k \rightarrow R^{N+k}$ defined by the composition

$$\overline{x}(s) = \phi_1^{s_1} \circ \phi_2^{s_2} \circ \cdots \circ \phi_k^{s_k}(\overline{x}_0) \tag{9}$$

where $\phi_i^s(\cdot)$, $i = 1, \cdots, k$ denotes the flow corresponding to a vector field γ_i on ES, $\{\gamma_i, i = 1, \cdots, k\}$ is a set of smooth vector fields that span $\ker D_{\overline{x}}F$, and $\overline{x}_0$ is any distinguished point of ES.

Proof: This is essentially the sufficiency part of the Frobenius theorem as presented in [5, Sec. 1.4] and [4, pp. 39–41].

B. Example 2

Consider the two-parameter uncontrolled family

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1^3 - \mu_1 x_1 - \mu_0 \end{bmatrix}. \tag{10}$$

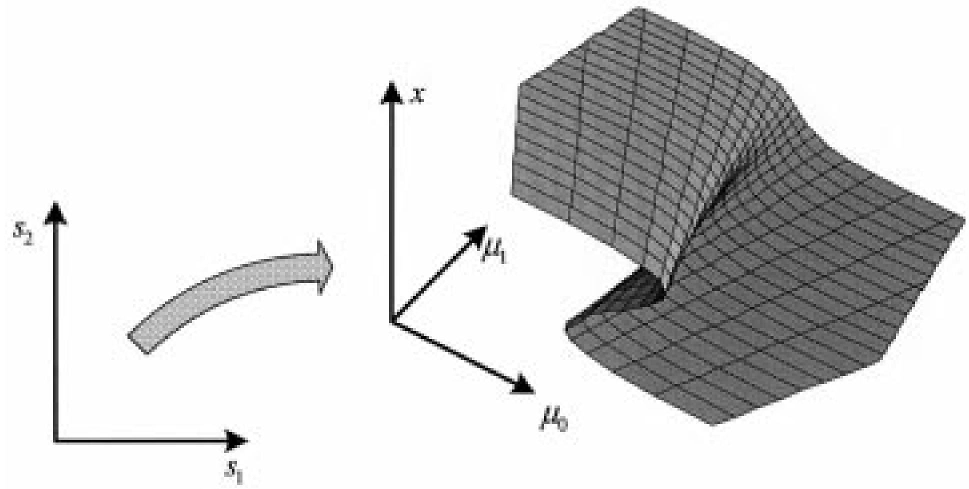

Fig. 3. This figure illustrates the mapping defined in Example 2. The two-dimensional parameter space (s-space) generates the two-dimensional equilibrium surface imbedded in the three-dimensional (x, μ_0, μ_1) space.

The equilibrium surface is defined by $x_2 = 0$ and

$$F(x_1, \mu_0, \mu_1) := x_1^3 + \mu_1 x_1 + \mu_0 = 0.$$

Since x_2 is trivially determined, the surface is defined by this equation. Thus

$$D_{\overline{x}}F = [3x_1^2 + \mu_1 \quad 1 \quad x_1]$$

and a basis for ker $D_{\overline{x}}F$ is

$$\gamma_1(x_1, \mu_0, \mu_1) = \begin{bmatrix} 1 \\ -3x_1^2 - \mu_1 \\ 0 \end{bmatrix}, \qquad \gamma_2(x_1, \mu_0, \mu_1) = \begin{bmatrix} 0 \\ -x_1 \\ 1 \end{bmatrix}.$$

The equations to be solved are

$$\frac{\partial}{\partial s_1}\begin{bmatrix} x_1 \\ \mu_0 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3x_1^2 - \mu_1 \\ 0 \end{bmatrix}, \qquad \frac{\partial}{\partial s_2}\begin{bmatrix} x_1 \\ \mu_0 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \\ 1 \end{bmatrix}$$

which generate the flows

$$\phi_1(x_1, \mu_0, \mu_1) = \begin{bmatrix} s_1 + x_1 \\ -s_1^3 - 3s_1^2 x_1 - 3s_1 x_1^2 - \mu_1 s_1 + \mu_0 \\ \mu_1 \end{bmatrix}$$

$$\phi_2(x_1, \mu_0, \mu_1) = \begin{bmatrix} x_1 \\ -x_1 s_2 + \mu_0 \\ \mu_1 + s_2 \end{bmatrix}.$$

We will place the origin of the new coordinates at the point $(x_1, \mu_0, \mu_1) = (0, 0, 0) \in ES$ so that the desired mapping is

$$\overline{x}(s_1, s_2) = \phi_1(\phi_2(0, 0, 0)) = \phi_2(\phi_1(0, 0, 0)) = \begin{bmatrix} s_1 \\ -s_1^3 - s_1 s_2 \\ s_2 \end{bmatrix}. \tag{11}$$

The mapping (11) is illustrated in Fig. 3.

It is easy to compute the perturbation dynamics via

$$A = \frac{\partial f}{\partial x}(x_0, \mu)\bigg|_{(x_0, \mu) \to \overline{x}(s)}.$$

Thus, we obtain

$$A = \begin{bmatrix} 0 & 1 \\ -3x_{1,0}^2 - \mu_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3s_1^2 - s_2 & 0 \end{bmatrix}. \tag{12}$$

We easily see from (12) that the unstable region is defined by $3s_1^2 + s_2 < 0$. This region can be mapped onto the equilibrium surface from which we find that the "wedge"-shaped region within the folds contains the unstable equilibria.

Note that the Jacobian $\partial\mu/\partial s$ must be singular at bifurcation points of the equilibrium equations. Hence, in this example, we compute

$$\frac{\partial \mu}{\partial s} = \begin{bmatrix} -3s_1^2 - s_2 & -s_1 \\ 0 & 1 \end{bmatrix}$$

to obtain the condition

$$\left|\frac{\partial \mu}{\partial s}\right| = -3s_1^2 - s_2 = 0.$$

As expected in this simple example, the (static) bifurcation points correspond to the (divergence) stability boundary.

C. Flows via the Exponential Map

Consider the differential equation

$$\frac{dx}{ds} = \gamma(x), \qquad x(0) = x_0. \tag{13}$$

We can write the solution trajectory as a Taylor series in s about $s = 0$

$$x(s) = x_0 + \dot{x}(0)s + \frac{1}{2!}\ddot{x}(0)s^2 + \frac{1}{3!}\dddot{x}(0)s^3 \cdots$$

or

$$x(s) = x_0 + \gamma(x_0)s + \frac{1}{2!}L_\gamma[\gamma](x_0)s^2 + \frac{1}{3!}L_\gamma^2[\gamma](x_0)s^3 + \cdots.$$

So, provided the series converges, the flow can be expressed as follows:

$$\phi^s(x) = x + \sum_{k=0}^{\infty} \frac{1}{(k+1)!} L_\gamma^k[\gamma](x)s^{k+1}. \tag{14}$$

Equation (14) provides the flow in the form of the "exponential map" associated with the vector field γ [4], [6]. It is in a form that is easily computable to any desired order in s.

D. Computer Implementation

We have implemented the required computations in *Mathematica*, building on the functions described in [7]. The calculations involve four steps:

1) computing the Jacobian DF;
2) generating a smooth basis set for ker DF;

3) computing the flow functions based on the exponential map (14);
4) forming the composition (9).

The only subtlety is Step 2), in which case care must be taken to ensure that a smooth set of basis vectors is generated. The *Mathematica* function `NullSpace` typically does not return a smooth basis set. A smooth set, however, is desirable because it ensures that the remaining computations proceed efficiently.

Because we use the exponential map to compute the flow functions, our results are local. Even so, we are able to capture essential nonlinear behavior as shown in the examples.

IV. Longitudinal Flight Dynamics

As a more substantial example of the method described above, we will apply it to the longitudinal dynamics of an aircraft. The model, taken from [2], is

$$\begin{bmatrix} \cos a & -v\sin\alpha & v\sin\alpha & 0 \\ \sin a & v\cos\alpha & -v\cos\alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} v \\ \alpha \\ \theta \\ q \end{bmatrix} = \begin{bmatrix} -\sin\theta + \Lambda_w \sin\alpha + \Lambda_t \sin\alpha_t + T - \Delta\cos\alpha \\ \cos\theta - \Lambda_w\cos\alpha - \Lambda_t\cos\alpha_t - \Delta\sin\alpha \\ q \\ \dfrac{V_0^2 l^*}{gr^2}\{\Sigma_w + \kappa\Lambda_w\cos\alpha - (1-\kappa)\Lambda_t\cos\alpha_t\} - \dfrac{cV_0}{mgr^2}q \end{bmatrix}$$

where the states are: v—normalized speed; α—angle of attack; θ—pitch angle; and q—pitch rate. Other variables are: $\alpha_t = \alpha + \delta_e$—tail angle of attack; $\Lambda_w, \Lambda_t, \Delta, \Sigma_w$—normalized aerodynamic forces; T—normalized engine thrust; V_0—the cruise velocity; l^*, r—characteristic lengths; κ—the center of gravity location parameter ($\kappa > 0$ places the center of gravity behind its nominal location at the wing center of pressure); m—mass; and g—gravitational constant. For illustrative and computational purposes we complete the model by specifying the following functions and parameters:

$$\begin{aligned} \Lambda_w &= f_w(\alpha)\overline{\rho}v^2, \qquad \Lambda_t = f_t(\alpha_t)\overline{\rho}v^2 \\ \Delta &= \left(a + b[f_w(\alpha)]^2\right)\overline{\rho}v^2, \qquad \Sigma_w = \sigma_w(\alpha)\overline{\rho}v^2 \\ f_w(\alpha) &= \left[\alpha - 2.05(\alpha - 0.05)^3\right]/0.05 \\ f_t(\alpha_t) &= 0.1\left[(\alpha_t - 0.05) - 3(\alpha_t - 0.05)^3\right]/00.05 \\ \sigma_w(\alpha) &= 0, \; \overline{\rho} = 1, \qquad a = b = 0.05 \\ V_0^2 l^*/gr^2 &= 300, \qquad cV_0/mgr^2 = 8. \end{aligned}$$

Let $\gamma = \alpha - \theta$ denote the flight path angle. We seek to investigate the small perturbation behavior of the aircraft as it flies along a linear flight path corresponding to speed v^* and flight path angle γ^*. Consequently, the equilibrium point is defined by (15), as shown at the bottom of the page. With $\gamma^* = 0$ and treating κ, v^* as parameters, we can compute the transformation rules that define the mapping $(s_1, s_2) \to (v_0, \alpha_0, \theta_0, q_0, T_0, \delta_{e0}, \kappa, v^*)$ and hence the equilibrium surface. They are given in the Appendix. Fig. 4 illustrates one of the surfaces generated by the mapping. Once the equilibrium surface is characterized it is a simple matter to compute the $A(s)$, $B(s)$, $C(s)$, and $D(s)$ matrices. These are also given in the Appendix where, for convenience of presentation, we have expanded these matrices in (s_1, s_2) up to terms of order three.

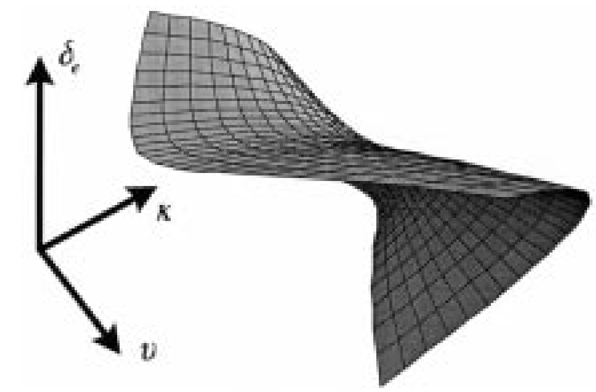

Fig. 4. This figure illustrates the surface that defines the relationship between elevator deflection, speed, and center of gravity location. As before, the surface is generated by first obtaining a parametric characterization of the equilibrium manifold.

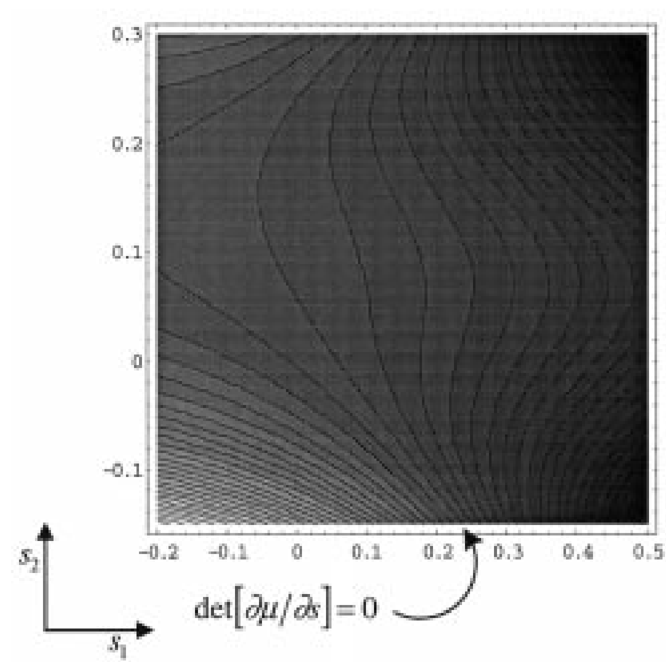

Fig. 5. This figure illustrates a contour plot of $\det[\partial\mu/\partial s]$. The fold bifurcation points correspond to the level curve $\det[\partial\mu/\partial s] = 0$. Points to the left of this curve map to the lower sheet of the surface in Fig. 5, and points to the right map to the upper sheet.

As the parameters vary, the linear system properties change. The fold bifurcation points are associated with a transmission zero located at the origin (see [2]). This property of control systems is different from that of dynamical systems (ordinary differential equations,

$$\begin{aligned} f(x, u, \mu) &= \begin{bmatrix} -\sin\theta + \Lambda_w \sin\alpha + \Lambda_t \sin\alpha_t + T - \Delta\cos\alpha \\ \cos\theta - \Lambda_w\cos\alpha - \Lambda_t\cos\alpha_t - \Delta\sin\alpha \\ q \\ \dfrac{V_0^2 l^*}{gr^2}\{\Sigma_w + \kappa\Lambda_w\cos\alpha - (1-\kappa)\Lambda_t\cos\alpha_t\} - \dfrac{cV_0}{mgr^2}q \end{bmatrix} = 0 \\ h(x, u, \mu) &:= \begin{bmatrix} v - v^* \\ \alpha - \theta - \gamma^* \end{bmatrix} = 0 \end{aligned} \tag{15}$$

without inputs and outputs). In the latter case the fold bifurcation points correspond to a pole at the origin, as in Examples 1 and 2. Once again, we can identify the bifurcation values for the s-parameters by computing those points for which the Jacobian $\partial\mu/\partial s$ is singular. For the present example, this is illustrated in Fig. 5.

V. CONCLUSIONS

This paper has considered the construction of a parameter-dependent linear family of perturbation dynamics for a control system described by a parameter-dependent set of nonlinear state and output equations. Such families are used in the design of gain-scheduled, robust, and adaptive controllers. A method for constructing them is proposed that is based on a globally valid reparameterization of the equilibrium surface. Symbolic computing tools have been implemented that enable the efficient assembly of local families. Local families can be constructed around any equilibrium point, including bifurcation points of the equilibrium equations. We have presented some examples including the construction of a linear, parameter-dependent family describing the longitudinal dynamics of an aircraft.

VI. APPENDIX
COMPUTATIONAL RESULTS FOR SECTION IV

Transformation Rules:

$$
\begin{aligned}
&\{\text{kappa} \to -0.684403s1^3 + (1.72367s2 - 1.05951)s1^2 \\
&\quad + 8.12991(s2 - 0.250651)(s2 - 0.0763583)s1 \\
&\quad - 2.67297(s2 - 0.282496) \\
&\quad \cdot (s2^2 + 0.166192s2 + 0.123174) \\
&VO \to 0.280373s1^3 + (0.375957 - 1.59897s2)s1^2 \\
&\quad + (1.1085s2 - 6.20337s2^2 - 0.0556018)s1 \\
&\quad - 24.0763(s2 - 0.328683) \\
&\quad \cdot (s2^2 + 0.0641432s2 + 0.0850141), \\
&\text{alpha0} \to 1.s2 + 0.2,\ \text{theta0} \to 1.s2 + 0.2 \\
&\quad q \to 0,\ T0 \to 1.(\sin(1.s2 + 0.2) - 0.5(0.280373)s1^3 \\
&\quad + (0.375957 - 1.59897s2)s1^2 \\
&\quad + (1.1085s2 - 6.20337s2^2 - 0.0556018)s1 \\
&\quad - 24.0763(s2 - 0.328683) \\
&\quad \cdot (s2^2 + 0.0641432s2 + 0.0850141))^2 \\
&\quad \cdot (-0.05(400.(1.s2 - 2.05(1.s2 + 0.15)^3 + 0.2)^2 + 1.) \\
&\quad \cdot \cos(1.s2 + 0.2) + 20.(1.s2 - 2.05(1.s2 + 0.15)^3 + 0.2) \\
&\quad \cdot \sin(1.s2 + 0.2) \\
&\quad + 2.(1.s1 + 1.s2 - 3.(1.s1 + 1.s2 + 0.25)^3 + 0.25) \\
&\quad \cdot \sin(1.s1 + 1.s2 + 0.3)) \\
&\text{dele0} \to 1.s1 + 0.1\}.
\end{aligned}
$$

A, B, C, D Matrices:

$$
\begin{aligned}
&A(s_1,\ s_2) \\
&\{\{0,\ 1,\ 0,\ 0\},\ \{0,\ 8.,\ 0,\ 115.44 + 1507.s1 - 972.08s1^2 \\
&\quad - 521.228s1^3 - 414.698s2 - 10035.6s1s2 \\
&\quad + 18693.8s1^2s2 + 4613.63s2^2 + 79831.3s1s2^2 \\
&\quad - 26665.8s2^3\} \\
&\{-1.,\ 0,\ -0.507965 + 0.372692s1 - 0.0413567s1^2 \\
&\quad - 3.53916s1^3 - 3.25592s2 - 1.24722s1s2 \\
&\quad - 8.59549s1^2s2 - 0.19869s2^2 - 3.18888s1s2^2 \\
&\quad + 20.569s2^3,\ -0.520796 + 0.267623s1 - 4.13749s1^2 \\
&\quad - 4.93026s1^3 + 2.78097s2 - 7.69581s1s2 \\
&\quad + 10.0341s1^2s2 + 12.3308s2^2 + 30.1295s1s2^2 \\
&\quad - 14.9802s2^3\} \\
&\{0,\ 1.,\ -4.26585 - 0.830071s1 + 4.76698s1^2 \\
&\quad + 5.94933s1^3 - 18.1031s2 + 9.04239s1s2 \\
&\quad + 17.4469s1^2s2 + 22.9275s2^2 + 17.4608s1s2^2 \\
&\quad + 46.97s2^3,\ -6.34348 + 3.73129s1 + 2.19369s1^2 \\
&\quad - 5.20817s1^3 + 27.7291s2 - 7.14437s1s2 \\
&\quad + 6.69362s1^2s2 - 45.6417s2^2 + 77.3689s1s2^2 \\
&\quad +265.795s2^3\}\}.
\end{aligned}
$$

$$
\begin{aligned}
&B(s_1,\ s_2) \\
&\{\{0,\ 0\},\ \{0, -88.1598 + 1200.04s1 \\
&\quad + 1174.58s1^2 + 59.6655s1^3 + 1553.95s2 \\
&\quad - 2588.05s1s2 - 2091.08s1^2 - 5328.94s2^2 \\
&\quad + 18061.s1s2^2 + 26256.4s2^3\} \\
&\{0.980067 - 0.198669s2 - 0.490033s2^2 \\
&\quad + 0.0331116s2^3,\ 0.111244 + 0.16315s1 \\
&\quad - 3.42676s1^2 - 4.46184s1^3 - 0.515347s2 \\
&\quad - 5.30715s1s2 + 4.78682s1^2s2 + 1.2976s2^2 \\
&\quad + 15.8889s1s2^2 - 7.7875s2^3\} \\
&\{-0.295306 - 0.0244064s1 + 0.163009s1^2 \\
&\quad + 0.150181s1^3 - 2.13244s2 + 0.254494s1s2 \\
&\quad + 0.92401s1^2s2 - 1.93552s2^2 + 1.44401s1s2^2 \\
&\quad + 5.43439s2^3,\ -0.279218 + 3.2301s1 \\
&\quad + 5.58259s1^2 - 2.68087s1^3 + 3.6805s2 \\
&\quad + 3.68856s1s2 - 13.4051s1^2s2 \\
&\quad - 3.72915s2^2 + 7.89261s1s2^2 \\
&\quad +25.149s2^3\}\}
\end{aligned}
$$

$$
\begin{aligned}
&C(s_1,\ s_2) \\
&\{\{0,\ 0,\ 1,\ 0\},\ \{-1,\ 0,\ 0,\ 1\}\}
\end{aligned}
$$

$$
\begin{aligned}
&D(s_1,\ s_2) \\
&\{\{0,\ 0\},\ \{0,\ 0\}\}.
\end{aligned}
$$

REFERENCES

[1] W. S. Levine, *The Control Handbook.* Boca Raton, FL: CRC, 1996.

[2] H. G. Kwatny, W. H. Bennett, and J. M. Berg, "Regulation of relaxed stability aircraft," *IEEE Trans. Automat. Contr.,* vol. 36, pp. 1325–1323, 1991.

[3] R. F. Stengel and W. B. Nixon, "Stalling characteristics of general aviation aircraft," *J. Aircraft,* vol. 19, pp. 425–434, 1982.

[4] P. J. Olver, *Applications of Lie Groups to Differential Equations.* New York: Springer-Verlag, 1986.

[5] A. Isidori, *Nonlinear Control Systems,* 3rd ed. London, U.K.: Springer-Verlag, 1995.

[6] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups.* New York: Springer-Verlag, 1983.

[7] G. L. Blankenship, R. Ghanadan, H. G. Kwatny, C. LaVigna, and V. Polyakov, "Integrated tools for modeling and design of controlled nonlinear systems," *IEEE Contr. Syst.,* vol. 15, pp. 65–79, 1995.